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Minimal Pure Injective Resolutions of Flat Modules

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Every module has a minimal pure injective resolution. For a flat modul over a noetherian ring, the structure of the pure injective modules appearing in such a resolution is now known and can be used to give information about minimal pure injective resolutions of flat modules. This information can in turn be used to give new proofs of results about the projective dimension of flat modules and at the same time sharpen these results. A change of ring theorem gives the perhaps surprising result that the pure injective dimensions of the coordinate rings of affine algebraic varieties over some fixed ground field are the same for all varieties of a given dimension n.

NOTATION

R will always denote a commutative noetherian ring. If R is local, $m(R)$ will denote its maximal ideal. For a prime ideal $\mathfrak{p} \subset R$, $k(\mathfrak{p})$ will denote the quotient field of $R/\mathfrak{p}$. Module will mean R-module, unless otherwise specified. For any module M, $\hat{M}_\mathfrak{p}$ will denote the separated completion of $M_\mathfrak{p}$ with the $m(R_\mathfrak{p})$-adic topology.

For a prime ideal $\mathfrak{p}$, the coheight of $\mathfrak{p}$ will mean the same as the depth of $\mathfrak{p}$ as defined by Nagata [10]. For any module M, $\mathrm{PE}(M)$ will denote a pure injective envelope of M (this was shown to exist by Fuchs [13]). M is a pure submodule of $\mathrm{PE}(M)$, $\mathrm{PE}(M)$ is pure injective, and if $S \subset \mathrm{PE}(M)$ with $M \cap S = 0$, then $M + S/S$ pure in $\mathrm{PE}(M)/S$ implies $S = 0$. Equivalently, M is a pure submodule of the pure injective module. $\mathrm{PE}(M)$, and any endomorphism of $\mathrm{PE}(M)$ which is the identity on M is an automorphism of $\mathrm{PE}(M)$.

If $\phi\colon M \to P$ is any injection into a pure injective module P such that $\phi(M)$ is pure in M (in the language of [2], ϕ is a pure injective pre-envelope of M), then the diagram

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "i"'] & P \arrow[d, dotted, "f"] \\ & \mathrm{PE}(M) \end{tikzcd}$$

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can be completed to a commutative diagram. It is an easy argument then that any such f admits a section, so $\mathrm{PE}(M)$ is isomorphic to a direct summand of P. The sequence $0 \to M \to \mathrm{PE}^0(M) \to \mathrm{PE}^1(M) \to \cdots$ will denote a minimal pure injective resolution of M. $\mathrm{E}(M)$ will denote an injective envelope of M. If X is a set, M^X will denote the module of functions $X \to M$, and $M^{(X)}$ the submodule of functions with finite support.

$\mathrm{Hom}(M, N)$ will mean $\mathrm{Hom}_R(M, N)$, and similarly $M \otimes N$ will denote $M \otimes_R N$ unless otherwise specified.

Recall that for any prime ideal $\mathfrak{p} \subset R$, by Matlis [3] we have $\hat{R}_\mathfrak{p} \simeq \mathrm{Hom}(\mathrm{E}(k(\mathfrak{p})), \mathrm{E}(k(\mathfrak{p})))$. We will let $T_\mathfrak{p}$ stand for the completion of some free $R_\mathfrak{p}$-module with the $m(R_\mathfrak{p})$-adic topology. As noted in [4, Proposition 1.3], $T_\mathfrak{p}$ can be realized as $\mathrm{Hom}(\mathrm{E}(k(\mathfrak{p})), \mathrm{E}(k(\mathfrak{p}))^{(X)})$ for some set X indexing the base of the free module whose completion is $T_\mathfrak{p}$.

We note that if E is an injective module, $\mathrm{Hom}(M, E)$ is pure injective [4, Lemma 2.1].

The projective dimension and pure injective dimension of M will be denoted proj. dim. M and pure inj. dim. M when R is understood.

1. Pure Injective Envelopes of Flat Modules

For completeness we prove

LEMMA 1.1 (Gruson and Jensen [11]). *If G is a flat module, then $\mathrm{PE}(G)$ is flat.*

Proof. Warfield [1] showed that the canonical map $G \to \mathrm{Hom}_Z(\mathrm{Hom}_Z(G, Q/Z), Q/Z)$ is an injection and maps G onto a pure submodule (i.e., is a pure injection). Since G is flat, $\mathrm{Hom}_Z(G, Q/Z)$ is injective. But then by Ishikawa [5, Theorem 1.5], $\mathrm{Hom}_Z(\mathrm{Hom}_Z(G, Q/Z), Q/Z)$ is flat. As noted above, $\mathrm{PE}(G)$ is a direct summand of $\mathrm{Hom}_Z(\mathrm{Hom}_Z(G, Q/Z), Q/Z)$ and so is flat.

Flat cotorsion modules (i.e., flat modules F such that $\mathrm{Ext}^1(G, F) = 0$ for all flat modules G) were characterized in [4, Theorem] as products $\prod T_\mathfrak{p}$ (all prime ideals $\mathfrak{p} \subset R$) where $T_\mathfrak{p}$ is the completion of a free $R_\mathfrak{p}$-module. Furthermore, $\prod T_\mathfrak{p}$ is uniquely determined by the dimensions of the free $R_\mathfrak{p}$-modules whose completions are the $T_\mathfrak{p}$ (or equivalently, by the dimension $T_\mathfrak{p} \otimes k(\mathfrak{p})$ over $k(\mathfrak{p})$). Since pure injective modules are cotorsion, the module $\mathrm{PE}(F)$ is such a module whenever F is flat. (In fact, a flat module is cotorsion if and only if it is pure injective.)

We will need

LEMMA 1.2 (Raynaud and Gruson [6, Proposition 2.4.3.1]). *For any flat module F, $\hat{F}_\mathfrak{p}$ is the completion of a free $R_\mathfrak{p}$-module.*

We note that $F \to \hat{F}_{\mathfrak{p}}$ induces an isomorphism $F \otimes k(\mathfrak{p}) \to \hat{F}_{\mathfrak{p}} \otimes k(\mathfrak{p})$. Furthermore, $F \to \hat{F}_{\mathfrak{p}}$ is a universal map into such a module, i.e.,

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & \hat{F}_{\mathfrak{p}} \arrow[d, dotted] \\ & T \end{tikzcd}$$

can be completed uniquely to a commutative diagram whenever T is the completion of a free $R_{\mathfrak{p}}$-module.

Note that if $\mathfrak{p}F_{\mathfrak{p}} = F_{\mathfrak{p}}$ (or equivalently if $F_{\mathfrak{p}} \otimes k(\mathfrak{p}) = 0$), then $\hat{F}_{\mathfrak{p}} = 0$.

Now for each prime ideal $\mathfrak{p}$, let $T_{\mathfrak{p}}$ be the completion of a free $R_{\mathfrak{p}}$-module. Then for any $\mathfrak{p}$ we have

LEMMA 1.3. $\operatorname{Hom}(\prod_{\mathfrak{p} \not\subset \mathfrak{q}} T_{\mathfrak{q}}, T_{\mathfrak{p}}) = 0$.

Proof. We can assume $T_{\mathfrak{p}} = \operatorname{Hom}(\mathrm{E}(k(\mathfrak{p})), \mathrm{E}(k(\mathfrak{p}))^{(X)})$ for some set X. By a natural identity we need to show that

$$\operatorname{Hom}\left(\prod_{\mathfrak{q} \not\subset \mathfrak{p}} T_{\mathfrak{q}} \otimes \mathrm{E}(k(\mathfrak{p})), \mathrm{E}(k(\mathfrak{p}))^{(X)}\right) = 0;$$

hence, we only need argue that $\prod_{\mathfrak{q} \not\subset \mathfrak{p}} T_{\mathfrak{q}} \otimes \mathrm{E}(k(\mathfrak{p})) = 0$. If $S \subset \mathrm{E}(k(\mathfrak{p}))$ is finitely generated (as an R-module), then $(\Pi T_{\mathfrak{q}}) \otimes S \simeq \prod (T_{\mathfrak{q}} \otimes S)$. But for $\mathfrak{q} \not\subset \mathfrak{p}$, let $r \in \mathfrak{q}$, $r \notin \mathfrak{p}$. Then $r^n S = 0$ for some $n \geqslant 1$. Since $r^n \colon T_{\mathfrak{q}} \to T_{\mathfrak{q}}$ is an isomorphism, $T_{\mathfrak{q}} \otimes S = 0$ and so $(\prod T_{\mathfrak{q}}) \otimes S = 0$. Taking an inductive limit we get $(\prod T_{\mathfrak{q}}) \otimes \mathrm{E}(k(\mathfrak{p})) = 0$.

COROLLARY 1. $\operatorname{Hom}(\hat{R}_{\mathfrak{p}}, \hat{R}_{\mathfrak{q}}) \neq 0$ *if and only if* $\mathfrak{p} \supset \mathfrak{q}$.

Proof. We only need argue that if $\mathfrak{p} \supset \mathfrak{q}$, then $\operatorname{Hom}(\hat{R}_{\mathfrak{p}}, \hat{R}_{\mathfrak{q}}) \neq 0$. Since $\hat{R}_{\mathfrak{q}} \simeq \operatorname{Hom}(\mathrm{E}(k(\mathfrak{q})), \mathrm{E}(k(\mathfrak{q})))$, by a natural identity we only need argue that $\operatorname{Hom}(\hat{R}_{\mathfrak{p}} \otimes \mathrm{E}(k(\mathfrak{q})), \mathrm{E}(k(\mathfrak{q}))) \neq 0$. But $R_{\mathfrak{p}} \to \hat{R}_{\mathfrak{p}}$ is a pure injection [1, Theorem 3] as $R_{\mathfrak{p}}$-modules and so as R-modules, so $R_{\mathfrak{p}} \otimes \mathrm{E}(k(\mathfrak{q})) \to \hat{R}_{\mathfrak{p}} \otimes \mathrm{E}(k(\mathfrak{q}))$ is injective. Since $\mathfrak{q} \subset \mathfrak{p}$, $R_{\mathfrak{p}} \otimes \mathrm{E}(k(Q)) \simeq \mathrm{E}(k(\mathfrak{q}))$. Hence, $\hat{R}_{\mathfrak{p}} \otimes E(k(\mathfrak{q}))$ contains a copy of $\mathrm{E}(k(\mathfrak{q}))$ and so $\operatorname{Hom}(\hat{R}_{\mathfrak{p}} \otimes \mathrm{E}(k, (\mathfrak{q})), \mathrm{E}(k(\mathfrak{q}))) \neq 0$. (In fact, the argument shows $\phi \in \operatorname{Hom}(\hat{R}_{\mathfrak{p}}, \hat{R}_{\mathfrak{q}})$ can be found with $\phi(1) = 1$.)

Note that if F is flat, then $F \otimes \mathrm{E}(k(\mathfrak{q}))$ is injective. In fact it is not hard to see that it is a direct sum of copies of $\mathrm{E}(k(\mathfrak{q}))$. This implies $\operatorname{Hom}(F, \hat{R}_{\mathfrak{q}}) \simeq \hat{R}_{\mathfrak{q}}^X$ for some set X. If $F = \hat{R}_{\mathfrak{p}}$ with $\mathfrak{p} \subset \mathfrak{q}$, then $\operatorname{Card}(X) = 1$ if and only if $R_{\mathfrak{p}} \otimes \mathrm{E}(k(Q)) \to \hat{R}_{\mathfrak{p}} \otimes \mathrm{E}(k(\mathfrak{q}))$ is an isomorphism.

COROLLARY 2. *For any set X, $\hat{R}_{\mathfrak{q}}^X$ is the completion of a free $R_{\mathfrak{p}}$-module and every such completion is a direct summand of $\hat{R}_{\mathfrak{p}}^X$ for some X.*

Proof. $\hat{R}_\mathfrak{p}$ is cotorsion and flat, hence so is $\hat{R}_\mathfrak{p}^X$. So by [2, Theorem], $\hat{R}_\mathfrak{p}^X \simeq \prod T_\mathfrak{q}$ where each $T_\mathfrak{q}$ is the completion of a free $R_\mathfrak{q}$-module. If $T_\mathfrak{q} \neq 0$ then easily $\operatorname{Hom}(T_\mathfrak{q}, \hat{R}_\mathfrak{p}) \neq 0$, so by the lemma, $\mathfrak{q} \supset \mathfrak{p}$. If $r \notin \mathfrak{p}$ then multiplication by r on $\hat{R}_\mathfrak{p}^X$ is an isomorphism, and so it must be an isomorphism on each $T_\mathfrak{q}$. But if $T_\mathfrak{q} \neq 0$ then $r \notin \mathfrak{q}$ must hold. So $T_\mathfrak{q} \neq 0$ only if $\mathfrak{q} = \mathfrak{p}$, i.e., $\hat{R}_\mathfrak{p}^X \simeq T_\mathfrak{p}$. Now let $T_\mathfrak{p}$ be the completion of any free $R_\mathfrak{p}$-module. Assume $T_\mathfrak{p} = \operatorname{Hom}(\mathrm{E}(k(\mathfrak{p})), \mathrm{E}(k(\mathfrak{p}))^{(X)})$. Since $\mathrm{E}(k(\mathfrak{p}))^{(X)}$ is a direct summand of $\mathrm{E}(k(\mathfrak{p}))^X$, $T_\mathfrak{p}$ is isomorphic to a direct summand of $\operatorname{Hom}(\mathrm{E}(k(\mathfrak{p})), \mathrm{E}(k(\mathfrak{p}))^X) \simeq \hat{R}_\mathfrak{p}^X$.

PROPOSITION 1.1. *If F is a flat module, then the natural map $F \to \prod \hat{F}_\mathfrak{p}$ is a pure injection.*

Proof. Let $\mathrm{PE}(F) = \prod T_\mathfrak{p}$ as above. Then for each $\mathfrak{p}$

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & \hat{F}_\mathfrak{p} \arrow[d, dotted] \\ & T_\mathfrak{p} \end{tikzcd}$$

can be completed to a commutative diagram (using the obvious map $F \to T_\mathfrak{p}$). This gives a map $f\colon \prod \hat{F}_\mathfrak{p} \to \prod T_\mathfrak{p}$ such that

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & \prod F_\mathfrak{p} \arrow[d, "f"] \\ & \prod T_\mathfrak{p} = \mathrm{PE}(F) \end{tikzcd}$$

is commutative. Since $F \to \mathrm{PE}(F)$ is a pure injection, so is $F \to \prod \hat{F}_\mathfrak{p}$.

Using the same notation, we have

PROPOSITION 1.2. *For each $\mathfrak{p}$, $T_\mathfrak{p}$ is isomorphic to a direct summand of $\hat{F}_\mathfrak{p}$. If $\mathfrak{p}$ is maximal such that $F \otimes k(P) \neq 0$, then the map $\hat{F}_\mathfrak{p} \to T_\mathfrak{p}$ is an isomorphism.*

Proof. Since $F \to \prod \hat{F}_\mathfrak{p}$ is a pure injection, $f\colon \prod \hat{F}_\mathfrak{p} \to \mathrm{PE}(F) = \prod T_\mathfrak{p}$ has a section s making

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & \prod \hat{F}_\mathfrak{p} \\ & \prod T_\mathfrak{p} \arrow[u, "s"'] \end{tikzcd}$$

commutative. This guarantees that each $\hat{F}_\mathfrak{p} \to T_\mathfrak{p}$ has a section, say $s_\mathfrak{p}$.

(Note that in general $s \neq \prod s_{\mathfrak{p}}$, and in fact examples can be constructed so that no section s making the diagram above commutative is such a product.) And so $T_{\mathfrak{p}}$ is isomorphic to direct summand of $\hat{F}_{\mathfrak{p}}$. Note that if $F_{\mathfrak{p}} \otimes k(\mathfrak{p}) = 0$, then $\hat{F}_{\mathfrak{p}} = 0$ and so $T_{\mathfrak{p}} = 0$.

Now let $\mathfrak{p}$ be maximal such that $F \otimes k(\mathfrak{p}) \neq 0$. Then $\hat{F}_{\mathfrak{p}} \neq 0$ but $\hat{F}_{\mathfrak{q}} = 0$ if $\mathfrak{p} \subsetneq \mathfrak{q}$, so by Lemma 1.3, $\operatorname{Hom}(\prod_{\mathfrak{q} \neq \mathfrak{p}} \hat{F}_{\mathfrak{q}}, T_{\mathfrak{p}}) = 0$. Similarly, $\operatorname{Hom}(\prod_{\mathfrak{q} \neq \mathfrak{p}} T_{\mathfrak{q}}, \hat{F}_{\mathfrak{p}}) = 0$ since $T_{\mathfrak{q}} = 0$ whenever $\hat{F}_{\mathfrak{q}} = 0$.

Now consider the commutative diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[dr] \arrow[ddr] & \prod \hat{F}_{\mathfrak{q}} = \prod_{\mathfrak{q} \neq \mathfrak{p}} \hat{F}_{\mathfrak{q}} \oplus \hat{F}_{\mathfrak{p}} \arrow[d, "f"] \\
 & \prod T_{\mathfrak{q}} = \prod_{\mathfrak{q} \neq \mathfrak{p}} T_{\mathfrak{q}} \oplus T_{\mathfrak{p}} \arrow[d, "s"] \\
 & \prod_{\mathfrak{q} \neq \mathfrak{p}} \hat{F}_{\mathfrak{q}} \oplus \hat{F}_{\mathfrak{p}}
\end{tikzcd}$$

Using the above, we can pass to the quotients and get a commutative diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[dr] \arrow[ddr] & \hat{F}_{\mathfrak{p}} \arrow[d] \\
 & T_{\mathfrak{p}} \arrow[d] \\
 & \hat{F}_{\mathfrak{p}}
\end{tikzcd}$$

The composition $\hat{F}_{\mathfrak{p}} \to T_{\mathfrak{p}} \to \hat{F}_{\mathfrak{p}}$ is the identity on $\hat{F}_{\mathfrak{p}}$. Since $\hat{F}_{\mathfrak{p}} \to \hat{T}_{\mathfrak{p}}$ has a section, it must be an isomorphism.

Remark. Warfield [1, Theorem 3] shows that when $F = R$, $R \to \prod \hat{R}_{\mathfrak{m}}$ (for all maximal ideals $\mathfrak{m}$ of R) is a pure injective envelope. So with the notation above, $T_{\mathfrak{p}} = 0$ for evere $\mathfrak{p}$ which is not maximal. If F is free, say $F = R^{(X)}$. Then $R^{(X)} \subset R^X$ is pure, and $R^X \to (\prod \hat{R}_{\mathfrak{m}})^X \simeq \prod(\hat{R}^X_{\mathfrak{m}})$ is a pure injection, and so $R^{(X)} \to \prod(\hat{R}^X_{\mathfrak{m}})$ is a pure injection. By Corollary 2 to Lemma 1.3, $\hat{R}^X_{\mathfrak{m}}$ is the completion of a free $R_{\mathfrak{m}}$-module. Hence for any free module F, and so for any projective module F, $\mathrm{PE}(F) = \prod \hat{F}_{\mathfrak{m}}$.

If $F = R \oplus \hat{R}_{\mathfrak{p}}$ where P is not maximal, then if $\mathrm{PE}(F) = \prod T_{\mathfrak{q}}$, then $T_{\mathfrak{p}} = \hat{R}_{\mathfrak{p}}$, $T_{\mathfrak{m}} = \hat{R}_{\mathfrak{m}}$ for all maximal $\mathfrak{m}$ and $T_{\mathfrak{q}} = 0$ otherwise.

2. Minimal Pure Injective Resolutions of Flat Modules

Proposition 2.1. *If F is flat, then for each $n \geqslant 0$,* $\mathrm{PE}^n(F)$ *is flat.*

Proof. By Lemma 1.1, $\mathrm{PE}^0(F) = \mathrm{PE}(F)$ is flat. Since $F \subset \mathrm{PE}(F)$ is flat, so $C = \mathrm{PE}(F)/F$ is flat. Since $\mathrm{PE}^1(F) = \mathrm{PE}(C)$, Lemma 1.1 implies $\mathrm{PE}^1(F)$ (and by induction $\mathrm{PE}^n(F)$ for $n \geqslant 1$) is flat.

The following definition is analogous to Bass' definition of $\mu_n(\mathfrak{p}, M)$ [7].

Definition. For a flat module F and a prime ideal $\mathfrak{q}$ and $n \geqslant 0$, $\pi_n(\mathfrak{q}, M)$ is the cardinality of a base of a free $R_\mathfrak{q}$-module whose completion is where $\mathrm{PE}^n(F) \simeq \prod T_\mathfrak{q}$ and each $T_\mathfrak{q}$ is the completion of a free $R_\mathfrak{q}$-module.

Theorem 2.1. *If F is a flat module and a prime ideal $\mathfrak{P} \subset R$ is such that $\pi_n(\mathfrak{Q}, F) = 0$ for all $\mathfrak{Q} \supsetneqq \mathfrak{P}$, then $\pi_{n+1}(\mathfrak{Q}, F) = 0$ for all $\mathfrak{Q} \supset \mathfrak{P}$ (including $\mathfrak{Q} = \mathfrak{P}$).*

Proof. We give the argument for $n = 0$ as it is easy to modify for $n \geqslant 0$. Let $PE(F) = \prod T_\mathfrak{Q}$. Then by Proposition 1.2 we know that $T_\mathfrak{P} \cong \hat{F}^\mathfrak{P}$. By assumption $T_\mathfrak{Q} = 0$ if $\mathfrak{Q} \supsetneqq \mathfrak{P}$. We argue that $k(\mathfrak{P}) \otimes \prod_{\mathfrak{Q} \neq \mathfrak{P}} T_\mathfrak{q} = 0$. It suffices to argue that $S \otimes \prod_{\mathfrak{Q} \neq \mathfrak{P}} T_\mathfrak{Q} \cong \prod_{\mathfrak{Q} \neq \mathfrak{P}} (S \otimes T_\mathfrak{Q}) = 0$ when $S \subset k(\mathfrak{P})$ is finitely generated. But if $\mathfrak{Q} \supsetneqq \mathfrak{P}$, $T_\mathfrak{Q} = 0$ and if not, $rS = 0$ for any $r \in P$, $r \notin \mathfrak{Q}$, so easily $S \otimes T_\mathfrak{Q} = 0$.

Now let $0 \to F \to \mathrm{PE}(F) \to C \to 0$ be exact. Since $\mathrm{PE}(F) \otimes k(\mathfrak{p}) \simeq \hat{F}_\mathfrak{p} \otimes k(\mathfrak{p})$ by the above, $F \otimes k(\mathfrak{p}) \to \mathrm{PE}(F) \otimes k(\mathfrak{p})$ is an isomorphism. Hence $C \otimes k(P) = 0$. If $\mathfrak{q} \supsetneqq \mathfrak{p}$ then an argument as above gives $\mathrm{PE}(F) \otimes k(\mathfrak{q}) = 0$ and so $C \otimes k(\mathfrak{q}) = 0$. Since $C \otimes k(\mathfrak{q}) = 0$ implies $C_\mathfrak{q} = 0$, Proposition 1.2 completes the proof.

We note that the theorem implies that if $\pi_n(\mathfrak{p}, F) \neq 0$, then coheight $\mathfrak{p} \geqslant n$. This immediately gives

Corollary (Gruson and Jensen [11, Theorem 7.1]). *If* $\dim R < \infty$ *then $PF^n(F) = 0$ for $n >$* $\dim R$.

This says the pure injective dimension of any flat module is at most $\dim R$. Also if $\dim R = n < \infty$, then $\pi_n(\mathfrak{p}, F) \neq 0$ implies that $\mathfrak{p}$ is a minimal prime ideal of R.

3. Applications

As noted in Raynaud and Gruson [6], $\mathrm{Ext}^n(G, F)$ for flat modules G and F can be computed using projective resolutions of G or pure injective resolutions of F (in the language of Enochs and Jenda [8], $\operatorname{Hom}(-,-)$ is balanced by $\text{Flat} \times \text{Pure Inj}$). So the corollary to Theorem 2.1 gives

PROPOSITION 3.1 (Raynaud and Gruson [6, Corollary 3.2.7]). *For any flat module* G, proj. dim. $G \leqslant \dim R$.

As in [6], let $d = \sup \text{proj. dim.}\, G$ (for all flat modules G). Suppose $d < \infty$. Then for some flat module F, $\operatorname{Ext}^d(G, F) \neq 0$ and so $\mathrm{PE}^d(F) \neq 0$. Let $\mathrm{PE}^d(F) = \prod T_\mathfrak{p}$. Let $\mathfrak{p}$ be minimal such that $T_\mathfrak{p} \neq 0$. Then $T_\mathfrak{p}$ has $\hat{R}_\mathfrak{p}$ as a direct summand. The corresponding injection in $\operatorname{Hom}(\hat{R}_\mathfrak{p}, \mathrm{PE}^d(G))$ clearly gives a non-zero element of $\operatorname{Ext}^d(\hat{R}_\mathfrak{p}, G)$. Let the map $\hat{R}_\mathfrak{p} \to \mathrm{PE}^d(G)$ be restricted to $R_\mathfrak{p} \simeq \mathrm{PE}^d(G)$. We claim this map gives a non-zero element of $\operatorname{Ext}^d(R_\mathfrak{p}, G)$, for if

$$\begin{tikzcd} & R_\mathfrak{p} \arrow[dl] \arrow[d] \\ \mathrm{PE}^{d-1}(G) \arrow[r] & \mathrm{PE}^d(G) \end{tikzcd}$$

is commutative, then $R_\mathfrak{p} \to \mathrm{PE}^{d-1}(G)$ can be extended to $\hat{R}_\mathfrak{p} \to \mathrm{PE}^{d-1}(G)$. By the minimality of $\mathfrak{p}$, the extension of $R_\mathfrak{p} \to \mathrm{PE}^d(G)$ to $\hat{R}_\mathfrak{p}$ is unique by Lemma 1.3 and we would contradict the above.

This gives

COROLLARY 1. $d = \sup_\mathfrak{p} \text{proj. dim.}_R\, R_\mathfrak{p}$.

This strengthens Raynaud and Gruson [6, Theorem 3.3.1], who prove that $d = \sup \text{proj. dim.}_R\, S^{-1}R$ (taken over all multiplicative sets $S \subset R$).

COROLLARY 2. *If* $\mathfrak{p}$ *is minimal such that* $d = \text{proj. dim.}\, R_\mathfrak{p}$, *then* $d \leqslant$ *coheight* $\mathfrak{p}$.

Proof. This follows easily from the remark following the proof of Theorem 2.1 and the proof of Proposition 3.1.

It seems possible that $d = \sup \text{proj. dim.}\, R_\mathfrak{p}$ taken over all minimal prime ideals $\mathfrak{p}$. This hypothesis is suggested by Raynaud and Gruson when they ask if proj. dim. $F \leqslant$ proj. dim. K holds for all flat modules F when R is a domain and K its field of fractions. As noted after the corollary to Proposition 1.2, $\pi_n(F, \mathfrak{p}) \neq 0$ for $n = \dim. R < \infty$ means $\mathfrak{p}$ is minimal, so if R is a domain, $\mathfrak{p} = (0)$. This gives

COROLLARY 3. *If* R *is a domain with field of fractions* K *and* proj. dim. $K < \dim R < \infty$, *then* proj. dim. $F < \dim R$ *for all flat modules* F.

It is easy to construct examples with proj. dim. $K < \dim R$. In fact, if K is countably generated as an R-module, then proj. dim. $K \leqslant 1$.

Remark. As noted above, $\mathrm{PE}(R) = \prod \hat{R}_{\mathfrak{m}}$ which means $\prod_0(\mathfrak{m}, R) = 1$ for all maximal ideals $\mathfrak{m}$ and $\prod_0(\mathfrak{p}, R) = 0$ otherwise. From the above, $\pi_1(\mathfrak{m}, R) = 0$ for all such M. We can show for $\mathfrak{p}$ of coheight 1, $\pi_1(\mathfrak{p}, R) = 0$ if and only if $\mathfrak{p}$ is contained in a unique maximal ideal M and if $\widehat{(\hat{R}_{\mathfrak{m}})_{\mathfrak{p}}} \simeq \hat{R}_{\mathfrak{p}}$ holds. This has the consequence that for any R of pure injective dimension 1, any prime ideal $\mathfrak{p}$ of coheight 1 of $R[[X]]$ such that $X \notin \mathfrak{p}$ is contained in a unique such maximal ideal of $R[[X]]$.

I know of no counterexamples to the following:

(1) If $\mathfrak{p} \subset \mathfrak{q}$ are prime ideals of R, then proj. dim. $R_{\mathfrak{p}} \geqslant$ proj. dim. $R_{\mathfrak{q}}$.

(2) proj. dim. $R_{\mathfrak{p}} \leqslant$ coheight $\mathfrak{p}$ for each prime ideal $\mathfrak{p}$ of R, R local.

(3) $d = \sup$ proj. dim. $R_{\mathfrak{p}}$ taken over all minimal prime ideals $\mathfrak{p}$ of R.

(4) proj. dim. $R_{\mathfrak{p}} \leqslant$ proj. dim. $\hat{R}_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of R.

4. A Change of Ring Theorem

LEMMA 4.1. *If $\mathfrak{a} \subset R$ is an ideal and P is a flat pure injective module, then $\mathfrak{a}P$ is pure injective.*

Proof. If $P = \prod T_{\mathfrak{p}}$ then $\mathfrak{a}P \simeq \prod \mathfrak{a} T_{\mathfrak{p}}$. Then using Corollary 2 to Lemma 1.3, we only need establish that $\mathfrak{a}\hat{R}_{\mathfrak{p}}$ is pure injective. Since $\mathfrak{a}\hat{R}_{\mathfrak{p}}$ is reflexive as an $\hat{R}_{\mathfrak{p}}$-module, an application of Lemma 2.1 of [4] gives the result.

Note that for any $T_{\mathfrak{p}}$, $T_{\mathfrak{p}} \to T_{\mathfrak{p}} \otimes k(\mathfrak{p})$ is surjective with kernel $\mathfrak{p} T_{\mathfrak{p}} = m(\hat{R}_{\mathfrak{p}})\, T_{\mathfrak{p}}$. By the lemma this means that if F is a flat module, then any map $F \to T_{\mathfrak{p}} \otimes k(\mathfrak{p})$ can be lifted to a map $F \to T_{\mathfrak{p}}$. In fact, in [4] it was established that $\phi\colon T_{\mathfrak{p}} \to T_{\mathfrak{p}} \otimes k(\mathfrak{p})$ is a flat cover. In particular this means that if $f\colon T_{\mathfrak{p}} \to T_{\mathfrak{p}}$ is such that $\phi \circ f = \phi$, then f is an automorphism of $T_{\mathfrak{p}}$. A quick diagram chase then gives that any map $g\colon T_{\mathfrak{p}} \to T_{\mathfrak{p}}$ is an automorphism of $T_{\mathfrak{p}}$ if and only if the induced map $T_{\mathfrak{p}} \otimes k(\mathfrak{p}) \to T_{\mathfrak{p}} \otimes k(\mathfrak{p})$ is an isomorphism.

If V is any vector space over $k(\mathfrak{p})$, then for some $T_{\mathfrak{p}}$, $T_{\mathfrak{p}} \otimes k(\mathfrak{p}) \simeq V$; hence we have a map $T_{\mathfrak{p}} \to V$ with the lifting property above and which induces an isomorphism $T_{\mathfrak{p}} \otimes k(\mathfrak{p}) \simeq V$. This will be used in the following.

LEMMA 4.2. *If F and G are flat modules, then a map $\sigma\colon F \to G$ is a pure injection if and only if $F \otimes k(\mathfrak{p}) \to G \otimes k(\mathfrak{p})$ is an injection for every prime ideal $\mathfrak{p}$.*

Proof. The condition is necessary, so suppose the condition holds. It suffices to prove that

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & G \arrow[d, dotted] \\ & \mathrm{PE}(F) \end{tikzcd}$$

can be completed to a commutative diagram since $F \to \mathrm{PE}(F)$ is a pure injection. We argue that any diagram

$$\begin{tikzcd} F \arrow[r] \arrow[dr] & G \arrow[d, dotted] \\ & P \end{tikzcd}$$

where P is flat and pure injective can be completed to a commutative diagram. Letting $P = \prod T_\mathfrak{p}$ and appealing to Corollary 2 to Lemma 1.3, we see that we only need argue that for any prime ideal $\mathfrak{p}$ and map $\phi: F \to \hat{R}_\mathfrak{p}$

$$\begin{tikzcd} F \arrow[r, "\sigma"] \arrow[dr, "\phi"'] & G \arrow[d, dotted] \\ & \hat{R}_\mathfrak{p} \end{tikzcd}$$

can be completed. If we tensor with $k(\mathfrak{p})$ we have a diagram

$$\begin{tikzcd} F \otimes k(\mathfrak{p}) \arrow[r] \arrow[dr] & G \otimes k(\mathfrak{p}) \arrow[d, dotted] \\ & \hat{R}_\mathfrak{p}/m(\hat{R}_\mathfrak{p}) \end{tikzcd}$$

Since the horizontal map is injective and we have vector spaces, this diagram can be completed to a commutative diagram. As noted above, the map $G \to \hat{R}_\mathfrak{p}/m(\hat{R}_\mathfrak{p})$ can be lifted to a map $f_0: G \to \hat{R}_\mathfrak{p}$. The map f_0 completes the original diagram to a commutative diagram modulo $m(\hat{R}_\mathfrak{p})$, i.e., $\phi - f_0 \circ \sigma$ has its image in $m(\hat{R}_\mathfrak{p})$.

Now repeat the procedure (using Lemma 4.1 with $\mathfrak{a} = m(R_\mathfrak{p})^2$) with the diagram

$$\begin{tikzcd} F \arrow[r, "\sigma"] \arrow[dr, "\phi - f_0 \circ \sigma"'] & G \arrow[d, dotted, "f_1"] \\ & m(\hat{R}_\mathfrak{p}) \end{tikzcd}$$

finding f_1 so that $\phi - \sigma \circ f_0 - \sigma \circ f_1$ has its image in $m(\hat{R}_{\mathfrak{p}})^2$. If by induction we have $f_0, f_1, \ldots, f_n$ so that

$$\begin{tikzcd} F \arrow[r] \arrow[dr, "\phi - \sum_{i=0}^{n-1} f_i \circ \sigma"'] & G \arrow[d, "f_n"] \\ & m(\hat{R}_{\mathfrak{p}})^n \end{tikzcd}$$

is commutative modulo $m(\hat{R}_{\mathfrak{p}})^{n+1}$ for each n, we let $f = \sum_{i=0}^{\infty} f_i \colon G \to \hat{R}_{\mathfrak{p}}$. Then $f \circ \sigma = \phi$ and the proof is complete.

Note that since F and G are flat, $F/\mathfrak{p}F$ and $G/\mathfrak{p}G$ are flat, so torsion free, $R/\mathfrak{p}$-modules. This means $F \otimes R/\mathfrak{p} \to G \otimes R/\mathfrak{p}$ is an injection if and only if $F \otimes k(\mathfrak{p}) \to G \otimes k(\mathfrak{p})$ is an injection.

We now want to give a necessary and sufficient condition that a map $F \to \prod T_{\mathfrak{p}}$ be a pure injective envelope of the flat module F. We let $X = \operatorname{Spec}(R)$. If $Y \subset X$ is any subset such that $\mathfrak{q} \subset \mathfrak{p} \in Y$ for prime ideals $\mathfrak{q}$ and $\mathfrak{p}$ implies $\mathfrak{q} \in Y$, by Lemma 1.3 we see that any map $f\colon \prod T_{\mathfrak{p}} \to \prod T_{\mathfrak{p}}$ maps $\prod_{\mathfrak{p} \in Y} T_{\mathfrak{p}}$ into $\prod_{\mathfrak{p} \in Y} T_{\mathfrak{p}}$; hence, f induces a map on the quotient $\prod_{\mathfrak{p} \in X} T_{\mathfrak{p}} / \prod_{\mathfrak{p} \in Y} T_{\mathfrak{p}} \simeq \prod_{\mathfrak{p} \in X - Y} T_{\mathfrak{p}}$ into itself. Hence any commutative diagram

$$\begin{tikzcd} & \prod\limits_{\mathfrak{p} \in X} T_{\mathfrak{p}} \arrow[d] \\ F \arrow[ur] \arrow[dr] & \\ & \prod\limits_{\mathfrak{p} \in X} T_{\mathfrak{p}} \end{tikzcd}$$

gives rise, passing to the quotients, to a commutative diagram

$$\begin{tikzcd} & \prod\limits_{\mathfrak{p} \in X - Y} T_{\mathfrak{p}} \arrow[d] \\ F \arrow[ur] \arrow[dr] & \\ & \prod\limits_{\mathfrak{p} \in X - Y} T_{\mathfrak{p}} \end{tikzcd}$$

Furthermore, if $F \to \prod T_{\mathfrak{p}}$ is a pure injective envelope, then the vertical maps in both diagrams must be isomorphisms. This will be used in the sequel where it will be obvious the set Y in question has the required property. Using the notation above, we prove

THEOREM 4.1. *$F \to \prod T_{\mathfrak{p}}$ is a pure injective envelope if and only if*

(a) *for each prime ideal $\mathfrak{q}$, $F \otimes k(\mathfrak{q}) \to \prod T_{\mathfrak{p}} \otimes k(\mathfrak{q})$ is an injection;*

(b) *for each prime ideal* $\mathfrak{q}$, *the image of* $F \otimes k(\mathfrak{q})$ *in* $(\prod T_{\mathfrak{p}}) \otimes k(\mathfrak{q}) = (T_{\mathfrak{q}} \otimes k(\mathfrak{q})) \oplus (\prod_{\mathfrak{p} \neq \mathfrak{q}}) \otimes k(\mathfrak{q})$ *contains* $(T_{\mathfrak{q}} \otimes k(\mathfrak{q})) \oplus 0$.

Proof. We first construct $\prod T_{\mathfrak{p}}$ and a map $F \to \prod T_{\mathfrak{p}}$ satisfying (a) and (b). We then argue that any pure injective envelope of F satisfies (a) and (b). Then we show that if $f: F \to \prod T_{\mathfrak{p}}$ satisfies (a) and (b), it is in fact a pure injective envelope. Once these are established, the proof of the theorem will be complete.

First we construct $f: F \to \prod T_{\mathfrak{p}}$ satisfying (a) and (b).

Let $X = \operatorname{Spec}(R)$. Let X_0 be the set of maximal ideals of R. Define X_α for any ordinal $\alpha > 1$ so that X_α is the set of maximal elements of $X - \bigcup_{\beta < \alpha} X_\beta$. Then well order each X_α. Using these orders we can well order X so that if $\mathfrak{p} \in X_\alpha$ and $\mathfrak{q} \in X_\beta$ and $\alpha < \beta$, then $\mathfrak{p} < \mathfrak{q}$. Hence the $\mathfrak{p} \in X$ can be indexed by all $\alpha < \lambda$ for some ordinal λ so that if $\beta < \alpha < \lambda$ then $\mathfrak{p}_\alpha \not\supset \mathfrak{p}_\beta$.

We construct $T_{\mathfrak{p}_\beta}$ and the map $F \to T_{\mathfrak{p}_\beta}$ by transfinite induction. We let $T_{\mathfrak{p}_0} = \hat{F}_{\mathfrak{p}_0}$ and let $F \to T_{\mathfrak{p}_0} = \hat{F}_{\mathfrak{p}_0}$ be the natural map. Having constructed $T_{\mathfrak{p}_\beta}$ and $F \to T_{\mathfrak{p}_\beta}$ for all $\beta < \alpha < \lambda$, consider $F \otimes k(\mathfrak{p}_\alpha) \to \prod_{\beta < \alpha} T_{\mathfrak{p}_\beta} \otimes k(\mathfrak{p}_\alpha)$. Let V be its kernel and let $T_{\mathfrak{p}_\alpha}$ be such that there is a surjection $T_{\mathfrak{p}_\alpha} \to V$ inducing an isomorphism $T_{\mathfrak{p}_\alpha} \otimes k(\mathfrak{p}_\alpha) \to V$. Composing $F \to F \otimes k(\mathfrak{p}_\alpha)$ with a projection $F \otimes k(\mathfrak{p}_\alpha) \to V$ we get a map $F \to V$ which, as noted earlier, can be lifted to a map $F \to T_{\mathfrak{p}_\alpha}$. Then by construction $F \to \prod_{\beta \leqslant \alpha} T_{\mathfrak{p}_\beta}$ is such that $F \otimes k(\mathfrak{p}_\alpha) \to \prod_{\beta \leqslant \alpha} T_{\mathfrak{p}_\beta} \otimes k(\mathfrak{p}_\alpha)$ is an injection, and its image contains $T_{\mathfrak{p}_\alpha} \otimes k(\mathfrak{p}_\alpha) \oplus 0$.

We now claim $F \to \prod_{\alpha < \lambda} T_{\mathfrak{p}_\alpha}$ satisfies (a) and (b). Clearly (a) is satisfied. For $\beta < \lambda$, note that $\prod_{\alpha < \beta} T_{\mathfrak{p}_\alpha} \otimes k(\mathfrak{p}_\beta) = 0$, since $\mathfrak{p}_\beta \not\subset \mathfrak{p}_\alpha$ for $\alpha > \beta$. This means $(\prod_{\alpha < \lambda} T_{\mathfrak{p}_\alpha}) \otimes k(\mathfrak{p}_\beta) \simeq (\prod_{\alpha \leqslant \beta} T_{\mathfrak{p}_\alpha}) \otimes k(\mathfrak{p}_\beta)$, so by the above, (b) is satisfied.

Now let $F \to \prod U_{\mathfrak{p}_\alpha}$ be a pure injective envelope (with each $U_{\mathfrak{p}_\alpha}$ the completion of a free $R_{\mathfrak{p}_\alpha}$-module). Since $F \to \prod T_{\mathfrak{p}_\alpha}$ satisfies (a), by Lemma 4.2 we get a map $f: \prod T_{\mathfrak{p}_\alpha} \to \prod U_{\mathfrak{p}_\alpha}$ making

$$\begin{tikzcd} & \prod T_{\mathfrak{p}_\alpha} \arrow[dd, "f"] \\ F \arrow[ur] \arrow[dr] & \\ & \prod U_{\mathfrak{p}_\alpha} \end{tikzcd}$$

commutative. Similarly, we get a map $g: \prod U_{\mathfrak{p}_\alpha} \to \prod T_{\mathfrak{p}_\alpha}$ making the obvious diagram commutative. Tensoring with $k(\mathfrak{p}_\beta)$ for some β and noting that $\prod_{\alpha > \beta} T_{\mathfrak{p}_\alpha} \otimes k(\mathfrak{p}_\beta) = 0$ (since $\mathfrak{p}_\beta \not\subset \mathfrak{p}_\alpha$ for $\beta < \alpha$), we see that $\prod_{\alpha < \lambda} T_{\mathfrak{p}_\alpha} \otimes k(\mathfrak{p}_\beta) = \prod_{\alpha \leqslant \beta} T_{\mathfrak{p}_\alpha} \otimes k(\mathfrak{p}_\beta)$. Hence we have the commutative diagram

$$\begin{tikzcd}[column sep=small]
 & T_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)\oplus\left(\prod_{\alpha<\beta} T_{\mathfrak{p}_\alpha}\right)\otimes k(\mathfrak{p}_\beta) \arrow[d] \\
F\otimes k(\mathfrak{p}_\beta) \arrow[ur] \arrow[r] \arrow[dr] & U_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)\oplus\left(\prod_{\alpha<\beta} U_{\mathfrak{p}_\alpha}\right)\otimes k(\mathfrak{p}_\beta) \arrow[d] \\
 & T_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)\oplus\left(\prod_{\alpha<\beta} T_{\mathfrak{p}_\alpha}\right)\otimes k(\mathfrak{p}_\beta)
\end{tikzcd}$$

Since $\operatorname{Hom}(T_{\mathfrak{p}_\beta}, \prod_{\alpha<\beta} U_{\mathfrak{p}_\alpha})=0$ and $\operatorname{Hom}(U_{\mathfrak{p}_\beta}, \prod_{\alpha<\beta} T_{\mathfrak{p}_\alpha})=0$, the vertical maps above map $T_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)$ into $U_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)$ and $U_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)$ into $T_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)$. Since by (b) there is a subspace V of $F\otimes k(\mathfrak{p}_\beta)$ mapped isomorphically onto $T_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)$, we easily see the composition $T_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)\to U_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)\to T_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)$ is the identity map.

Now reverse the roles of T and U in the diagram above. A similar argument, but now using the fact that $F\to\prod U_{\mathfrak{p}_\alpha}$ is a pure injective envelope, gives that $U_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)\to T_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)\to U_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)$ is an automorphism of $U_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)$. This means $T_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)\to U_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)$ is an isomorphism, and so $V\subset F\otimes k(\mathfrak{p}_\beta)$ is mapped isomorphically onto $U_{\mathfrak{p}_\beta}\otimes k(\mathfrak{p}_\beta)$ in $\prod_{\alpha<\lambda} U_{\mathfrak{p}_\alpha}\otimes k(\mathfrak{p}_\beta)$. This shows that any pure injective envelope $F\to\prod_{\alpha<\lambda} U_{\mathfrak{p}_\alpha}$ satisfies (b). It obviously satisfies (a).

We now argue that if $F\to\prod T_{\mathfrak{p}_\alpha}$ satisfies (a) and (b) above, it is a pure injective envelope of F. By (a) and Lemma 4.2, we know it is a pure injection. We only need argue that if $f\colon \prod T_{\mathfrak{p}_\alpha}\to\prod T_{\mathfrak{p}_\alpha}$ makes

$$\begin{tikzcd}
 & \prod T_{\mathfrak{p}_\alpha} \arrow[d, "f"] \\
F \arrow[ur] \arrow[dr] & \\
 & \prod T_{\mathfrak{p}_\alpha}
\end{tikzcd}$$

commutative, then it is an automorphism of $\prod T_{\mathfrak{p}_\alpha}$. Since $\prod_{\alpha<\lambda} T_{\mathfrak{p}_\alpha}=\varprojlim_\beta(\prod_{\alpha\leqslant\beta} T_{\mathfrak{p}_\alpha})$, it suffices to show inductively that when we pass to the quotients $\prod_{\alpha\leqslant\beta} T_{\mathfrak{p}_\alpha}$ for each β, we get an automorphism of $\prod_{\alpha\leqslant\beta} T_{\mathfrak{p}_\alpha}$. Recall that $T_{\mathfrak{p}_0}=\hat{F}_{\mathfrak{p}_0}$. Hence f induces the identity map on the quotient $T_{\mathfrak{p}_0}$. Now suppose $\beta<\lambda$ is a limit ordinal and that f induces an isomorphism $\prod_{\alpha\leqslant\gamma} T_{\mathfrak{p}_\alpha}\to\prod_{\alpha\leqslant\gamma} T_{\mathfrak{p}_\alpha}$ for all $\gamma<\beta$. Taking a projective limit, we get that f induces an isomorphic $\prod_{\alpha<\beta} T_{\mathfrak{p}_\alpha}\to\prod_{\alpha<\beta} T_{\mathfrak{p}_\alpha}$. But we know $\prod_{\alpha\leqslant\beta} T_{\mathfrak{p}_\alpha}\to\prod_{\alpha\leqslant\beta} T_{\mathfrak{p}_\alpha}$ maps $T_{\mathfrak{p}_\beta}$ into $T_{\mathfrak{p}_\beta}$, so to prove this is an isomorphism we only need argue that $T_{\mathfrak{p}_\beta}$ is mapped isomorphically onto $T_{\mathfrak{p}_\beta}$. But tensoring with

$k(\mathfrak{p}_\beta)$ and using (b), we easily see the map $T_{\mathfrak{p}_\beta} \otimes k(\mathfrak{p}_\beta) \to T_{\mathfrak{p}_\beta} \otimes k(\mathfrak{p}_\beta)$ is an isomorphism. This guarantees that $T_{\mathfrak{p}_\beta} \to T_{\mathfrak{p}_\beta}$ is an isomorphism.

If β is not a limit ordinal, say, $\beta = \gamma + 1$, then a similar argument shows that $\prod_{\alpha \leqslant \gamma} T_{\mathfrak{p}_\alpha} \to \prod_{\alpha \leqslant \gamma} T_{\mathfrak{p}_\alpha}$ is an isomorphism. This guarantees that $\prod_{\alpha \leqslant \beta} T_{\mathfrak{p}_\alpha} \to \prod_{\alpha \leqslant \beta} T_{\mathfrak{p}_\alpha}$ is, too. This completes the proof.

We now get

THEOREM 4.2. *If $\sigma: R \to R'$ is a ring homomorphism such that R' is a finitely generated R-module, then for any flat R-module F, $0 \to F \otimes R' \to \mathrm{PE}^0(F) \otimes R' \to \mathrm{PE}'(F) \otimes R' \to \cdots$ is a minimal pure injective resolution of the R'-module $F \otimes R'$.*

Proof. The sequence is easily pure exact. We now show that each $\mathrm{PE}^n(F) \otimes R'$ is a flat, pure injective R'-module. Clearly it is flat. Suppose $\mathrm{PE}^n(F) = \prod T_\mathfrak{p}$. Since$(\prod T_\mathfrak{p}) \otimes R' \simeq \prod (T_\mathfrak{p} \otimes R')$, we only need establish that each $T_\mathfrak{p} \otimes R'$ is pure injective. By Corollary 2 to Lemma 1.3, this reduces to showing that $\hat{R}_\mathfrak{p} \otimes R'$ is pure injective. But as is well known, this is $\hat{R}'_{\mathfrak{q}_1} \oplus \cdots \oplus \hat{R}'_{\mathfrak{q}_s}$ where $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ are the distinct primes lying over $\mathfrak{p}$, and is 0 if there is no such prime.

To establish minimality, it suffices to argue that for any flat R-module F, $F \otimes R' \to \mathrm{PE}(F) \otimes R'$ is a pure injective envelope of F.

By (a) of Theorem 4.1, for any $\mathfrak{p} \subset R$, we have $F \otimes k(\mathfrak{p}) \to \mathrm{PE}(F) \otimes k(\mathfrak{p})$ is an injection and in fact splits, so $F \otimes k(\mathfrak{p}) \otimes R' \to \mathrm{PE}(F) \otimes k(\mathfrak{p}) \otimes R'$ splits. But $k(\mathfrak{p}) \otimes R'$ is the direct sum of a finite number of local rings of dimension 0. If we go modulo the radical of $k(\mathfrak{p}) \otimes R'$, we still get an injection, and in fact, we get $F \otimes R' \otimes (k(\mathfrak{q}_1) \oplus \cdots \oplus k(\mathfrak{q}_s)) \to \mathrm{PE}(F) \otimes R' \otimes (k(\mathfrak{q}_1) \oplus \cdots \oplus k(\mathfrak{q}_s))$ where $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ are the prime ideals of R' lying over $\mathfrak{p}$ (possibly $s = 0$). This shows that $F \otimes R' \to \mathrm{PE}(F) \otimes R'$ satisfies the condition (a) of Theorem 4.1. The argument that (b) is satisfied is completely similar.

An immediate result is

COROLLARY 1. pure inj. $\dim._{R'} R' \leqslant$ pure inj. $\dim._R R$.

Proof. Let $F = R$.

Note if $R \subset R'$, then for any flat module G, $G \otimes R' = 0$ implies $G = 0$. In this case, we get equality in the corollary.

If $R' = R/\mathfrak{a}$, and if pure inj. dim. $R = n < \infty$, it is easy to see from the proof that inequality holds in the corollary if and only if $\mathfrak{a} \not\subset \mathfrak{p}$ for any prime ideal $\mathfrak{p}$ such that $\pi_n(\mathfrak{p}, R) \neq 0$ so that in fact equality does occur for $R' = R/\mathfrak{p}$ if $\mathfrak{p}$ is a suitably chosen prime ideal of R which can always be taken minimal.

These remarks allow us to prove

COROLLARY 2. *If k is a field and R_1 and R_2 are the coordinate rings* *of affine algebraic varieties over k, then* $\dim R_1 = \dim R_2$ *implies* pure inj. $\dim._{R_1} R_1 =$ pure inj. $\dim._{R_2} R_2$.

Proof. If R_1 and R_2 are domains, then the normalization theorem and the remark above show that each of R_1 and R_2 has the same pure injective dimension as the same polynomial algebra.

If R_1 is not a domain, then for some minimal prime ideal $\mathfrak{p}$ of R_1, $R_1/\mathfrak{p}$ and R_1 have the same pure injective dimensions over themselves, so it is easy to complete the proof.

If $R \subset R'$ and R' is a finitely generated R-module, it is well known that if R is the product of a finite number of complete local rings, then so is R'.

We drop the assumption that R is noetherian, and we have

COROLLARY 3. *If* $R \subset R'$, *if* R' *is a finitely generated R-module, and* R' *is the direct product of a finite number of complete, local noetherian rings, then so is* R.

Proof. By Eakin [12], R is noetherian. By Gruson and Jensen [11, Theorem 9.1], the pure injective dimension of R is 0 if and only if R is as described in the corollary, so we only need note R and R' have the same pure injective dimension.

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